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LETTER TO THE EDITOR

Laplace transforms of Airy functions via their integral definitions

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Abstract. The Laplace transforms of the functions $\mathrm{Ai}(\pm\eta)$ and $\mathrm{Bi}(-\eta)$ are obtained using the integral definitions of these functions and a correction made to a recent derivation of the results for $\mathrm{Ai}(-\eta)$ and $\mathrm{Bi}(-\eta)$.

There has been some interest in the Laplace transforms of Airy's functions in such areas as magnetohydrodynamics and surface physics. The transforms were obtained independently by Smith (1973) and Davison and Glasser (1982). In both cases the starting point was Airy's differential equation. In Smith (1973), it is stated that integration of Fourier–Bessel series has also been used.

It is the purpose of this letter to show that these transforms may be obtained from the integral definitions of the Airy functions. In the course of the derivation, an error in the results of Davison and Glasser (1982) is indicated. Considerable use is made of the standard works of Abramowitz and Stegun (1965) and Gradshteyn and Ryzhik (1980). To facilitate reference to a formula from one of these works it will be quoted as ASn and GRn where n refers to the number of the formula in the appropriate text.

The integral representations of the Airy functions are (AS 10.4.32,33)

$$\mathrm{Ai}(\pm\eta)=\frac{1}{\pi}\int_0^\infty \cos\left(\tfrac{1}{3}\zeta^3\pm\eta\zeta\right)\mathrm{d}\zeta \tag{1}$$

$$\mathrm{Bi}(\pm\eta)=\frac{1}{\pi}\int_0^\infty \left[\exp\left(\tfrac{1}{3}\zeta^3\pm\eta\zeta\right)+\sin\left(\tfrac{1}{3}\zeta^3\pm\eta\zeta\right)\right]\mathrm{d}\zeta. \tag{2}$$

Denoting the Laplace transform of $\mathrm{Ai}(\pm\eta)$ by $I_{\mathrm{A}}^{\pm}(p)$,

$$\begin{aligned} I_{\mathrm{A}}^{\pm}(p)&=\frac{1}{\pi}\int_0^\infty \exp(-p\eta)\,\mathrm{d}\eta\int_0^\infty \cos\left(\tfrac{1}{3}\zeta^3\pm\eta\zeta\right)\mathrm{d}\zeta\\ &=\frac{1}{\pi}\int_0^\infty \mathrm{d}\zeta\left(\cos\tfrac{1}{3}\zeta^3\int_0^\infty \exp(-p\eta)\cos\eta\zeta\,\mathrm{d}\zeta\right.\\ &\qquad\left.\mp\sin\tfrac{1}{3}\zeta^3\int_0^\infty \exp(-p\eta)\sin\eta\zeta\,\mathrm{d}\zeta\right). \end{aligned} \tag{3}$$

The inner integrals in (3) are the real and minus imaginary parts of

$$\int_0^\infty \exp[-\eta(p+\mathrm{i}\zeta)]\,\mathrm{d}\eta=(p+\mathrm{i}\zeta)^{-1} \tag{4}$$

respectively. Substitution of (4) in (3) together with the change of variable $\zeta = up$ and the replacement of $\frac{1}{3}p^3$ by a gives

$$\pi I_n^{\pm}(p) = \int_0^\infty (1+u^2)^{-1}(\cos au^3 \mp u \sin au^3)\,\mathrm{d}u. \tag{5}$$

The integral in (5) may be reduced to standard forms by substituting $u = \zeta^{1/3}$ and multiplying both numerator and denominator by $(1-\zeta^{2/3}+\zeta^{4/3})$:

$$3\pi I_{\mathrm{A}}^{\pm}(p) = \int_0^\infty (1+\zeta^2)^{-1}(\zeta^{-2/3}-1+\zeta^{2/3})(\cos a\zeta \mp \zeta^{1/3} \sin a\zeta)\,\mathrm{d}\zeta. \tag{6}$$

The integrals in (6) are given by GR3.766.1,2 for the fractional powers and by GR3.723.2,3 for the integral powers. (The former formulae require limits to be taken in the case of integral powers.) After some manipulation which uses the recurrence relations of the gamma and incomplete gamma functions

$$\Gamma(1+x) = x\Gamma(x) \tag{7}$$

$$a\gamma(a,x) = \gamma(1+a,x) + x^a\,\mathrm{e}^{-x} \tag{8}$$

we obtain

$$3\pi I_{\mathrm{A}}^{+}(p) = \mathrm{e}^{-a}[\pi + \tfrac{1}{2}\sqrt{3}\Gamma(\tfrac{1}{3})\gamma(\tfrac{2}{3},-a)\exp(2\pi\mathrm{i}/3) - \tfrac{1}{2}\sqrt{3}\Gamma(\tfrac{2}{3})\gamma(\tfrac{1}{3},-a)\exp(\pi\mathrm{i}/3)] \tag{9}$$

$$3\pi I_{\mathrm{A}}^{-}(p) = \mathrm{e}^{a}[2\pi - \tfrac{1}{2}\sqrt{3}\Gamma(\tfrac{1}{3})\gamma(\tfrac{2}{3},a) - \tfrac{1}{2}\sqrt{3}\Gamma(\tfrac{2}{3})\gamma(\tfrac{1}{3},a)]. \tag{10}$$

Since

$$\Gamma(\tfrac{1}{3})\Gamma(\tfrac{2}{3}) = 2\pi/\sqrt{3} \tag{11}$$

$$\gamma(\alpha,-x) = \exp(-\mathrm{i}\alpha\pi)\phi(\alpha,x) \tag{12}$$

where

$$\phi(\alpha,x) = \int_0^x \mathrm{e}^u u^{\alpha-1}\,\mathrm{d}u \tag{13}$$

we have

$$I_{\mathrm{A}}^{-}(p) = \tfrac{1}{3}\mathrm{e}^{a}[2-\gamma(\tfrac{2}{3},a)/\Gamma(\tfrac{2}{3}) - \gamma(\tfrac{1}{3},a)/\Gamma(\tfrac{1}{3})] \tag{14}$$

$$I_{\mathrm{A}}^{+}(p) = \tfrac{1}{3}\mathrm{e}^{-a}[1+\phi(\tfrac{2}{3},a)/\Gamma(\tfrac{2}{3}) - \phi(\tfrac{1}{3},a)/\Gamma(\tfrac{1}{3})]. \tag{15}$$

The Laplace transform of $\mathrm{Bi}(-\eta)$ is calculated in a similar fashion. The only real difference in the calculation is that the formulae GR3.723.2,3 are replaced by GR3.723.1,5 and the real exponential integral is evaluated using GR8.353.3. The result is that

$$I_{\mathrm{B}}^{-}(p) = (1/\sqrt{3})\,\mathrm{e}^{a}[-\gamma(\tfrac{2}{3},a)/\Gamma(\tfrac{2}{3}) + \Gamma(\tfrac{1}{3},a)/\Gamma(\tfrac{1}{3})]. \tag{16}$$

There is no Laplace transform for $\mathrm{Bi}(\eta)$ because of a singularity at $\zeta = p$. The formulae given in (14) and (16) differ from the corresponding formulae given by Davison and Glasser (1982). In their first term for $I_{\mathrm{B}}^{-}(p)$, there is no negative sign. This is possibly due to a misprint. In the $\gamma(\frac{1}{3},a)$ term of both (14) and (16), they give the sign opposite to that given here. This appears to be due to a misunderstanding of the meaning of the value of $\mathrm{Ai}'(-\eta)$ and $\mathrm{Bi}'(-\eta)$ at $\eta = 0$. As can be seen from a glance at figures 10.6,7 of AS, the gradients are positive and negative respectively and not the other way round.

References

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